β_n	is the depth of descent at n-th step along discrepancy gradient;
ω_1, ω_B, ω_A	are the residual terms of equations for finite increments;
t, τ	are the time;
τ_m	is the right-hand value of complete time interval;
q	is the heat flux.

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SOLUTION OF INVERSE COEFFICIENT PROBLEMS BY THE REGULARIZATION METHOD USING SPLINE FUNCTIONS

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The problem of determining the unknown coefficient in an equation of conservation of matter is discussed.

In a region $Q=\{(x, t): (0, 1) \times (0, 1)\}$ let the equation of conservation of matter be assigned in the form

$$\begin{gathered} Lu \equiv L_t u - aL_x^{(1)} u - \frac{da}{du} L_x^{(2)} u = f(x, t), \quad (x, t) \in Q, \\ u(x, 0) = \psi(x), \quad D_1 u = \varphi_1(t), \quad D_2 u = \varphi_2(t), \quad (x, t) \in \partial Q, \end{gathered} \tag{1}$$

where u(x, t) is the process under consideration; a(u) is an unknown coefficient; f(x, t) is a function of internal sources; $\psi(x)$, $\varphi_1(t)$, and $\varphi_2(t)$ are functions describing the initial and boundary conditions of the problem; L_t, $L_x^{(1)}$, and $L_x^{(2)}$ are differential operators expressing one or another conservation law; D_1 and D_2 are boundary-condition operators; ∂Q is the boundary of the region.

Within the framework of models with the simultaneous estimation of parameters the following formulations are known: first, when the coefficient is sought from an additional condition to the problem (1) [1, 2], and second, when it is sought from known δ-approximations to u and f, i.e., from elements $\hat{u}$ and $\hat{f}$ such that $\rho_U(u, \hat{u}) \le \delta_1$ and $\rho_F(f, \hat{f}) \le \delta_2$ [3]. The second formulation, although connected with a greater volume of measurements, still allows one to construct models of processes which are closer to the actual processes. We will have this formulation in mind below.

Following [4], we introduce the regularizing functional

$$\Phi_\alpha [a] = \iint_Q (L\hat{u} - \hat{f})^2 \, dxdt + \alpha \Omega_{p,q}^{(k)}, \tag{2}$$

where α is the regularization parameter; $\Omega_{p,q}^{(k)}$ is a stabilizer of the form

$$\Omega_{p,q}^{(k)} = \begin{cases} \iint_Q \left[p(a-a^*)^2 + \left(\frac{d^q a}{du^q} - \frac{d^q a^*}{du^q} \right)^2 \right] dxdt, & k=1, \\ \iint_Q \left[p\left(\frac{\partial^p a}{\partial x^p} - \frac{\partial^p a^*}{\partial x^p} \right)^2 + q\left(\frac{\partial^q a}{\partial t^q} - \frac{\partial^q a^*}{\partial t^q} \right)^2 \right] dxdt, & k=2, \end{cases} \tag{3}$$

where a^* is the trial element.

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The stabilizers introduced include the cases considered in [4-6] and impose the corresponding limitations on the region of allowable solutions, and consequently they must assure the regularity of the algorithm for the solution. The classes of stabilizers under consideration (k = 1, 2), however, differ in the efficiency of construction of the solution: for stabilizers of the second class (k = 2) there is regularization of both the functions $a(u)$ and $u(x, t)$, as a consequence of which one can expect that they possess better properties.

For the functional (2) let us examine the problem of seeking the coefficient a_α for which

$$\inf_a \Phi_\alpha[a] = \Phi_\alpha[a_\alpha]. \tag{4}$$

Since inverse problems are solved under conditions of limited information, as a rule, the unknown dependence must be chosen so that it proves to be less connected with a priori information, has sufficient smoothness, and assures good approximation. The enumerated requirements are filled by splines, the final form of which is determined by the desired properties of the stabilizer. In the present work we chose cubic splines [7]

$$a^{(l)}(u) = \lambda_1^{(l-1)} S_1^{(l)} + \lambda_2^{(l-1)} S_2^{(l)} + \lambda_1^{(l)} S_3^{(l)} + \lambda_2^{(l)} S_4^{(l)}, \quad u \in [u_{l-1}, u_l], \tag{5}$$

where

$$S_1^{(l)} = \frac{(u_l - u)^2 [2(u - u_{l-1}) + h_l]}{h_l^3}; \quad S_2^{(l)} = \frac{(u_l - u)^2 (u - u_{l-1})}{h_l^2};$$

$$S_3^{(l)} = \frac{(u - u_{l-1})^2 [2(u_l - u) + h_l]}{h_l^3}; \quad S_4^{(l)} = \frac{(u - u_{l-1})^2 (u - u_l)}{h_l^2};$$

$$h_l = u_l - u_{l-1}; \quad l = \overline{1, N};$$

N is the lattice parameter and $R = \{u_l\}_{l=\overline{0,N}}$; $\{\lambda_{1,2}^{(l)}\}$ are coefficients. The investigation of the properties of the stabilizers up to second order inclusively (p, q = 0, 1, 2) is provided for in this case.

Reducing the functional (2) to a function of the variables $\{\lambda_{1,2}^{(l)}\}$, we write the necessary condition for the the minimum of this function in the form of a system of linear algebraic equations

$$A\lambda = b, \tag{6}$$

where

$$a_{ij} = a_{ji} = \begin{cases} V_{i,j}^{(1)}, & i = 1, 2, \quad j = \overline{i, 4}; \\ V_{i,j}^{(l-1)} + V_{i,j}^{(l)}, & i = 2l-1, 2l, \quad j = \overline{i, 2l}, \quad l = \overline{2, N}; \\ V_{i,j}^{(l)}, & i = 2l-1, 2l, \quad j = 2l+1, 2l+2, \quad l = \overline{2, N}; \\ V_{i,j}^{(N)}, & i = 2N+1, 2N+2, \quad j = i, 2N+2; \\ 0, & i = 2l-1, 2l, \quad j = \overline{2l+3, 2N+2}, \quad l = \overline{1, N-1}; \end{cases}$$

$$b_i = \begin{cases} W_i^{(1)} & i = 1, 2; \\ W_i^{(l-1)} + W_i^{(l)} & i = \overline{3, 2N}; \\ W_i^{(N)} & i = 2N+1, 2N+2; \end{cases}$$

$$V_{ij}^{(m)} = \iint_{Q_m} \Big\{ F_{i-2(m-1)}^{(m)} F_{j-2(m-1)}^{(m)} + \frac{\alpha}{k} \big[H_{p,i-2(m-1)}^{(m)} H_{p,j-2(m-1)}^{(m)} + G_{q,i-2(m-1)}^{(m)} G_{q,j-2(m-1)}^{(m)} \big] \Big\} \, d\xi d\tau;$$

$$W_i^{(m)} = \iint_{Q_m} \Big\{ (L_\tau v - f) F_{i-2(m-1)}^{(m)} + \frac{\alpha}{k} \big[H_{p,i-2(m-1)}^{(m)} Y_p + G_{q,i-2(m-1)}^{(m)} Z_q \big] \Big\} \, d\xi d\tau;$$

$$Q_m = \{(\xi, \tau), \ u_{m-1} \leqslant v(\xi, \tau) \leqslant u_m\};$$

$$F_t^{(r)} = S_t^{(r)} L_\xi^{(1)} v + \frac{dS_t^{(r)}}{dv} L_\xi^{(2)} v;$$

$$H_{p,t}^{(r)} = \begin{cases} 0, & k = 1, \quad p = 0, 1, 2; \\ p \left[\frac{d^p S_t^{(r)}}{dv^p} \left(\frac{\partial v}{\partial \xi} \right)^p + \frac{dS_t^{(r)}}{dv} \frac{\partial^p v}{\partial \xi^p} \right], & k = 2, \quad p = 0, 1, 2; \end{cases}$$

$$G_{q,t}^{(r)} = \begin{cases} \dfrac{d^q S_t^{(r)}}{dv^q}, & k=1, \quad q=0,\ 1,\ 2; \\ q\left[\dfrac{d^q S_t^{(r)}}{dv^q}\left(\dfrac{\partial v}{\partial \tau}\right)^q + \dfrac{dS_t^{(r)}}{dv}\dfrac{\partial^q v}{\partial \tau^q}\right], & k=2,\ q=0,\ 1,\ 2; \end{cases}$$

$$Y_p = \begin{cases} 0, & k=1, \quad p=0,\ 1,\ 2; \\ p\dfrac{\partial^p a^*}{\partial \xi^p}, & k=2, \quad p=0,\ 1,\ 2; \end{cases}$$

$$Z_q = \begin{cases} \dfrac{d^q a^*}{dv^q}, & k=1, \quad q=0,\ 1,\ 2; \\ q\dfrac{\partial^q a^*}{\partial \tau^q}, & k=2, \quad q=0,\ 1,\ 2. \end{cases}$$

The solution of the system (6) determines the coefficients of the spline (5) for a given α. The best regularization parameter can be determined from the discrepancy principle:

$$\iint_Q (\hat{u} - u_\alpha)^2\, dxdt = \delta_1^2 + \delta_a^2, \tag{7}$$

where u_α is the solution of the direct problem (1) with the given α; δ_a is the error in the algorithm for the solution of the inverse and direct problems. The practical calculation of δ_a is difficult, but if one can guarantee an estimate $\delta_a \ll \delta_1$ (in accordance with the functional scheme for the model and the process, for example, and in the presence of stability of the algorithm for the solution), then Eq. (7) closes the solution of the stated problem of identification.

The principle of the quasioptimum parameter, consisting in the solution of the problem

$$\inf_\alpha \iint_Q (u_\alpha - u_{\rho\alpha})^2\, dxdt = \iint_Q (u_{\alpha_*} - u_{\rho\alpha_*})^2\, dxdt, \tag{8}$$

where $\rho \sim 1$ is a parameter of the iteration process, evidently leads to another means of determination of the best $\alpha = \alpha_*$. Closure with respect to the algorithm (8) does not require knowledge of the statistical characteristics of the process under study, which offers undoubted advantages.

Let us consider a model problem: For the function

$$\hat{u}(x,\ t) = t + x^2(1 + \varepsilon t) \tag{9}$$

and the model

$$\begin{gathered} \frac{\partial u}{\partial t} = a(u)\frac{\partial^2 u}{\partial x^2} \quad (x,\ t) \in (0,\ 1) \times (0,\ 1), \\ u(x,\ 0) = x^2, \\ u(0,\ t) = t, \quad u(1,\ t) = 1 + t \end{gathered} \tag{10}$$

of the process one is required to determine the dependence $a(u)$.

Since the exact solution ($\varepsilon = 0$, $\bar{a} = 0.5$) is known in the problem under consideration, the efficiency in the use of the stabilizers (3) which were introduced can be estimated in the following way: As the best value of the parameter $\alpha = \alpha_*$ for each stabilizer $\Omega^{(k)}_{p,q}$ we take that which is the solution of the problem

$$\inf_\alpha \iint_Q (a_\alpha - \bar{a})^2\, dxdt = \iint_Q (a_{\alpha_*} - \bar{a})^2\, dxdt = \sigma_1^2. \tag{11}$$

The given algorithm allows one to examine the properties of the stabilizers from the point of view of estimating the mean-square error of the solution of the inverse problem. The operator mean-square error

$$\sigma_2^2 = \iint_Q (Lu - f)^2\, dxdt \tag{12}$$

was determined in parallel. The degrees of the stabilizer (the variables p and q) were chosen from the condition of continuous differentiability of the original functions.

TABLE 1. Comparison of the Methods of Least Squares and of Regularization

		$\alpha=0$			$\Omega_{0,0}^{(1)}$			$\Omega_{0,1}^{(1)}$			$\Omega_{0,1}^{(2)}$		
$\varepsilon=0$	N	5	20	40	5	20	40	5	20	40	5	20	40
	$\sigma_1^2\times10^4$	85	50	39	79	50	39	12,9	0,58	0,22	12,9	0,58	0,22
	$\sigma_2^2\times10^4$	21	32	8,6	19	32	8,6	38,9	1,68	1,00	36,6	1,68	0,99
$\varepsilon=0{,}1$	N	5	21	40	5	21	40	5	21	40	5	21	40
	$\sigma_1^2\times10^4$	129	35	27	128	33	27	24,6	1,17	0,81	20,4	1,12	0,80
	$\sigma_2^2\times10^4$	365	15	6,19	363	15,7	5,8	1,03	4,94	4,68	66,9	4,88	4,69

		$\Omega_{1,0}^{(2)}$			$\Omega_{1,1}^{(2)}$			$\Omega_{2,1}^{(2)}$		
$\varepsilon=0$	N	5	20	40	5	20	40	5	20	40
	$\sigma_1^2\times10^4$	13,2	5,92	25,9	13,1	0,44	0,20	5,94	4,84	21,3
	$\sigma_2^2\times10^4$	28,8	3,97	1,96	28,6	1,39	0,99	7,09	3,81	2,80
$\varepsilon=0{,}1$	N	5	21	40	5	21	40	5	21	40
	$\sigma_1^2\times10^4$	17,1	2,41	25,5	14	0,91	0,77	6,64	8,64	25,9
	$\sigma_2^2\times10^4$	40,6	6,84	5,53	29	4,61	4,76	8,68	7,06	5,58

TABLE 2. Values of the Coefficient $a(u)$ Found with $\varepsilon = 0.1$ and N = 40

	u	0	0,05	0,25	0,5	0,75	1	1,25	1,5	2	2,1
$\alpha=0$	a	0,997	0,501	0,498	0,488	0,484	0,497	0,499	0,498	0,501	0,992
$\Omega_{1,1}^{(2)}$	a	0,495	0,499	0,499	0,488	0,485	0,497	0,500	0,498	0,497	0,588

As the results of numerical experiments show (Table 1), the use of stabilizers of nonzeroth degree considerably improves the solution of the inverse problem. The stabilizer $\Omega_{0,0}^{(1)}$ is close in its properties to the method of least squares; i.e., in the spline approximation of the coefficient $a(u)$ the zeroth regularization is insufficient. As was assumed above, among the stabilizers of the two classes the best proves to be that which includes the additional regularization of the initial data (k = 2). The results of the solution of the model problem (9)-(11) are improved in proportion to the increase in the degree of the stabilizer up to a value of p = q = 1, which determines the continuity of the derivative of the spline (5), and they also improve with an increase in the parameter N of the spline lattice. Further, the degree of smoothness of the initial data $\hat{u}(x, t)$ influences the regularization. First, the smoothness determines the degree of the stabilizer, and second, from a comparison of stabilizers of the same degree ($\Omega_{0,1}^{(2)}$ and $\Omega_{1,0}^{(2)}$) one can cnnclude that the regularization is sensitive to the variable which represents less smoothness.

An important result of the use of stabilizers is the improvement of the properties of the solution as $\varepsilon \to 0$ and with a large fixed N: In this case the method of least squares leads to an additional error in the solution and, for example, $\sigma_{1,2}^2|_{\varepsilon=0} > \sigma_{1,2}^2|_{\varepsilon=0.1}$ with N = 20 and N = 40; i.e., the method of least squares does not assure convergence of the solution as $\varepsilon \to 0$.

The smallest values of the coefficient for the case of $\varepsilon = 0.1$ and N = 40 are presented in Table 2. It is seen from it that the error at the boundary of the region proves to be the largest. This should be expected, since the spline (5) is not a coordinate function, but this error is also smoothed out in the case of regularization.

In conclusion, we note that the problem of identification for several unknown coefficients requires additional study.

NOTATION

x	is the spatial vector;
t	is the time coordinate;
u(x, t)	is the process under consideration;
$\hat{u}(x, t)$	is its approximation;
f(x, t)	is the source function;
f(x, t)	is the measured value;
ρ_U, ρ_F	are the metrics of the functional spaces;
$a(u)$	is the unknown coefficient;
$\psi(x)$, $\varphi_1(t)$, $\varphi_2(t)$	are the functions describing the initial and boundary conditions of the problem;
L_t, $L_x^{(1)}$, $L_x^{(2)}$, D_1, D_2	are the operators of initial–boundary problem;
α	is the regularization parameter;
$\Phi_\alpha[a]$	is the regularizing functional;
$\Omega_{p,q}^{(k)}$	is the stabilizer;
k	is the class of stabilizer;
p, q	are the degrees of stabilizer;
l	is the spline index;
$\lambda_1^{(l)}$, $\lambda_2^{(l)}$	are the spline coefficients;
N	is the spline grid parameter;
u_α	is the solution of the direct problem with the given α.

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